

Coarse Grained Quantum Dynamics

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ABSTRACT: We consider coarse graining a quantum system divided between short distance and long distance degrees of freedom, which are coupled by the Hamiltonian. Observations using purely long distance observables can be described by the reduced density matrix that arises from tracing out the short-distance observables. The dynamics of this density matrix is that of an open quantum system, and is nonlocal in time, on the order of some short time scale. We describe these dynamics in a model system with a simple hierarchy of energy gaps $\Delta E_{UV} > \Delta E_{IR}$, in which the coupling between high-and low-energy degrees of freedom is treated to second order in perturbation theory. We then describe the equations of motion under suitable time averaging, reflecting the limited time resolution of actual experiments, and find an expansion of the master equation in powers of $\Delta E_{IR}/\Delta E_{UV}$, in which the failure of the system to be Hamiltonian or even Markovian appears at higher orders in this ratio. We compute the evolution of the density matrix in two specific examples – coupled spins, and linearly coupled simple harmonic oscillators. Finally, we discuss the evolution of the density matrix using the path integral approach, computing the Feynman-Vernon influence functional for the IR degrees of freedom in perturbation theory, and argue that this influence functional is the correct analog of the Wilsonian effective action for this problem.

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1. Introduction

Consider an interacting scalar field theory

$$H = \frac{1}{2}\pi_\phi^2 + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 \quad (1.1)$$

defined on some lattice with spacing a , with m, λ the bare couplings. In this paper we will consider the following situation: we have a measuring device which directly couples to the bare variables ϕ , but has finite resolution in space and time. That is, if we write the field ϕ in the Schrödinger picture as

$$\phi(x) = \sum_{k=-\pi/a}^{\pi/a} \frac{1}{2\sqrt{\pi\omega(k)}} a_k e^{ik \cdot x} + \text{h.c.} , \quad (1.2)$$

then our measuring devices couple to a_k, a_k^\dagger for $|k| < \Lambda \ll \frac{\pi}{a}$, and record the time of the measurement with temporal accuracy δt .

The Hilbert space can be broken up into

$$\mathcal{H} = \mathcal{H}_{IR} \otimes \mathcal{H}_{UV} \quad (1.3)$$

where \mathcal{H}_{IR} is generated by a_k^\dagger for $|k| < \Lambda$ and \mathcal{H}_{UV} is generated by a_k^\dagger for $|k| > \Lambda$. Note that we are not breaking up the Hilbert space according to energy scale. Firstly, for an interacting theory, spatial momentum and energy will not be directly related. Secondly, we may be interested in high-energy objects made up of many low-energy quanta. After all, the physics of the sun is well-described by the standard model cutoff at a TeV, even though its total mass is of order 10^{54} GeV .¹

We imagine an experiment of the following form. Begin with the system in its exact ground state $|0\rangle$, and act on it with some infrared operator \mathcal{O}_{IR} . Let the resulting state evolve in time,

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar} H t} \mathcal{O}_{IR} |0\rangle . \quad (1.4)$$

Now compute the probability of measuring the IR degrees of freedom in some state $|a\rangle \in \mathcal{H}_{IR}$. We are not making any measurements in \mathcal{H}_{UV} , so we should sum the probabilities over all possible final states in \mathcal{H}_{UV} . The result is

$$\begin{aligned} P(a, t) &= \sum_{|u\rangle \in \mathcal{H}_{UV}} |\langle u | \langle a | e^{-\frac{i}{\hbar} H t} \mathcal{O}_{IR} | 0 \rangle|^2 \\ &= \text{tr} \mathbb{P}_a e^{-\frac{i}{\hbar} H t} \mathcal{O}_{IR} | 0 \rangle \langle 0 | \mathcal{O}_{IR}^\dagger e^{\frac{i}{\hbar} H t} \\ &= \text{tr}_{\mathcal{H}_{IR}} \mathbb{P}_a \rho_{IR}(t) \end{aligned} \quad (1.5)$$

where $\mathbb{P}_a = |a\rangle \langle a|$, and

$$\rho_{IR}(t) = \text{tr}_{\mathcal{H}_{UV}} \left[e^{-\frac{i}{\hbar} H t} |\psi(0)\rangle \langle \psi(0)| e^{\frac{i}{\hbar} H t} \right] \quad (1.6)$$

More generally, the expectation value at time t of measurements of A_{IR} acting on \mathcal{H}_{IR} is $\langle A \rangle = \text{tr} A \rho_{IR}(t)$.

¹One may, however, wish to restrict the Hilbert space to states with low energy *density* of order Λ^D , where D is the space-time dimension.

The goal of this work is to investigate the dynamics of ρ_{IR} . When $\lambda \neq 0$ in (1.1), $|0\rangle$ and $\mathcal{O}_{IR}|0\rangle$ will be highly entangled with respect to the decomposition (1.3) [1, 2]. Furthermore, \mathcal{H}_{IR} is an open quantum system. If $[H, \mathcal{O}_{IR}] \neq 0$, then the time evolution of $|\psi(t)\rangle$ will be nontrivial and involve excitations being transferred between \mathcal{H}_{IR} and \mathcal{H}_{UV} . The time evolution of ρ_{IR} will have non-Markovian memory effects which we wish to characterize. In addition, we wish to take into account the practical fact of finite clock accuracy via a time-averaging procedure and understand the non-Hamiltonian and non-Markovian effects after this averaging (see [3] for some discussion of this in a holographic context).

We will focus on the case

$$H = H_{IR} + H_{UV} + \lambda V \quad (1.7)$$

where V couples \mathcal{H}_{IR} , \mathcal{H}_{UV} . We take the spectrum of H_{IR} , H_{UV} to be characterized by level spacings ΔE_{IR} , ΔE_{UV} , respectively, with $\Delta E_{IR} \ll \Delta E_{UV}$. We will treat λV as a perturbation. For initially pure states, we compute the evolution of $\rho(t)$ to $\mathcal{O}(\lambda^2)$; for initially mixed states we compute the evolution to $\mathcal{O}(\lambda)$. In the former case, the evolution equation can be written in a form local in time (See [4–6] and references therein):

$$i\hbar \frac{d}{dt} \rho(t) = [H_{eff}, \rho(t)] + i \sum_{ij} h_{ij}(t) \left[L_i(t) \rho(t) L_j^\dagger(t) - \frac{1}{2} \left\{ L_j^\dagger(t) L_i(t), \rho(t) \right\} \right] \quad (1.8)$$

where $L_i(t)$ are certain operators. Here the effective Hamiltonian H_{eff} arises from processes which do not induce transitions of the ultraviolet degrees of freedom, while the terms involving L_i involve precisely those transitions. Eq. (1.8) reduces to the Kossakowski-Lindblad equation [7, 8] characteristic of time-independent quantum Markov processes when h can be diagonalized to a positive-definite matrix, and h and L are time-independent. We will find that even after suitable time-averaging, h_{ij} fails to satisfy the positivity condition characteristic of quantum Markov processes (see [4–6] for recent reviews and discussions of measures of non-Markovianity in open quantum systems), at least for the simple truncated models we work with here.

In the case that the UV and IR systems have a nice path integral prescription, we will also compute the density matrix in the path integral formalism, and relate the Feynman-Vernon influence functional [9] to (1.8). This will give some insight into the non-Markovian behavior of the system.

This approach is a close cousin to other “Wilsonian” approaches to low-energy, long-wavelength physics. However, we are considering a slightly different scenario, in which our experimental apparatus couples to bare rather than to renormalized quantities. To make this point clear, we will begin by reviewing both the path integral [10, 11] and Hamiltonian [12, 13] approaches to renormalization pioneered by Wilson. We will also discuss recent work on the holographic picture of Wilsonian renormalization developed in [3, 14, 15], which inspired this work.

We close by noting some interesting related work. The dynamics of the density matrix arising from coarse-graining a single-particle system with respect to position was studied in [16]. The use of entanglement between short- and long-wavelength degrees of freedom to explain thermalization and decoherence in heavy ion collisions was discussed in [17]. The notion of long-wavelength degrees of freedom as an open quantum system was also discussed in [18], which appeared in the late stages of the work reported here. There is some tension between the statements and assumptions in that paper, and the more detailed work reported here, as we will discuss this further in the conclusions. Finally, after the first draft of this work appeared, a relation to the RG equation for the density matrix describing the IR dynamics of jets in high-energy particle collisions was discussed in [19].

2. Varieties of Wilsonian renormalization

There are many standard discussions of Wilsonian renormalization, but they do not confront the issue of entanglement between the UV and the IR and its consequences. These treatments give very accurate descriptions of physical phenomena. Thus it may seem that UV-IR entanglement is irrelevant in talking about the dynamics of coarse-grained degrees of freedom. For this reason, in this section we will review the standard approaches to Wilsonian renormalization and discuss how the questions addressed by these approaches are different from the one that we described in the introduction.

2.1 Renormalization in path integral language

The textbook discussion of Wilsonian renormalization (cf. [11, 20]) begins with a Euclidean path integral

$$Z = \int D\phi(x) e^{-S_{\Lambda_0}[\phi]} \quad (2.1)$$

for a field theory with UV cutoff Λ_0 . One breaks up $\phi(x)$ into $\phi = \phi_s + \phi_f$, where the “slow” fields ϕ_s are supported on Euclidean momenta $|k| < \Lambda$ and the “fast” fields ϕ_f are supported on Euclidean momenta $\Lambda < |k| < \Lambda_0$. We coarse-grain the theory by integrating over ϕ_f to find

$$Z = \int D\phi_s e^{-S_{\Lambda}[\phi_s]} \quad (2.2)$$

For long-wavelength questions, we can work with this latter presentation.

When Z represents the partition function in a classical equilibrium statistical physics problem, the interpretation is clear: S_{Λ} will represent the spatially coarse-grained classical Hamiltonian of the system. If we want to compute equal-time correlators in the analytical continuation to real-time, we need to impose periodic boundary conditions in the integral over the high frequencies, while fixing the IR

modes on two sides of a cut in time [2]. This yields an effective action and an associated density matrix that can be used to compute equal time correlation functions. But when Z represents the quantum-mechanical vacuum-vacuum transition amplitude computed via Euclidean continuation, the decimation procedure above fixes both the initial and final state of the short-wavelength degrees of freedom. For the inclusive finite-time probabilities discussed in the introduction, this procedure is not appropriate, beyond the leading order in a Born-Oppenheimer approximation (see §2.4). Note that when describing scattering of initially well-separated particle states into final states of the same form, the interactions are effectively inoperative at early and late times, and the assumption that the short-distance modes are in their ground state is essentially correct. This standard treatment is designed to produce transition amplitudes between initial and final asymptotic states where only the low energy modes are excited. By contrast, we are interested in finite-time questions for states that have UV-IR entanglement, including the natural ground states of interacting theories, and states produced by the action of coarse-grained operators.

As we will discuss in §4, for the inclusive finite-time questions we are discussing in this paper, the decimation procedure is best applied to the real-time path integral developed by Feynman and Vernon [9] for density matrices. In this case, the analog of the Wilsonian effective action will include terms describable as a renormalized Hamiltonian, together with a nontrivial “influence functional” which encodes the time development of entanglement between the IR and UV degrees of freedom.

A further issue arises from the fact that higher-derivative interactions are generically induced. These will include terms that are functions of $\ddot{\phi}_s$ and higher derivatives still, arising from nonlocalities on the scale of the running cutoff.² In general, the Wilsonian action S_Λ will not have an interpretation as an action that can be derived from the Legendre transform of a Hamiltonian, without adding Stückelberg fields. Such a procedure amounts to adding the short-distance degrees of freedom back in. The interpretation of these higher-derivative terms is clear in the holographic picture of Wilsonian renormalization we will discuss below [3]: they reflect the fact that the IR degrees of freedom comprise an open quantum system, and that there are memory effects on the time scale of the UV dynamics.

2.2 Holographic Wilsonian Renormalization

The Wilsonian approach to renormalization has a candidate implementation in holographic theories [14, 15]. Recall the dual of a d -dimensional large- N conformal field theory (N could be the rank of a gauge group, or the central charge of a 2D CFT) is string- or M-theory in $AdS_{d+1} \times X$, where X is some space with constant positive curvature. For CFTs on $\mathbb{R}^{1,d-1}$, one considers a Poincaré patch of anti-de Sitter

²A related discussion, which partially inspired this paper, can be found in [2].

space, with coordinates

$$ds^2 = R^2 \frac{dr^2}{r^2} + \frac{r^2}{R^2} dx_d^2 \quad (2.3)$$

Here dx_d^2 is the flat metric on d -dimensional Minkowski space; R is the radius of curvature of AdS_d .

Refs. [14, 15] provide a proposal for “integrating out” degrees of freedom at short distances, in the large- N limit in which the bulk is well-approximated by quantum field theory in the AdS background. The cutoff Λ is associated with a definite radial coordinate, $r_\Lambda = R^2 \Lambda$. One breaks up the path integral over fields propagating on AdS_d into modes with $r > r_\Lambda$ and $r < r_\Lambda$; interprets the path integral for $r < r_\Lambda$ with fixed fields at $r = r_\Lambda$ as the generating functions of correlators in the cutoff theory; and integrates this over the field values at $r = r_\Lambda$ weighted by the path integral over the fields for $r > r_\Lambda$.

In this procedure, nontrivial operators are induced at the cutoff *even when the theory is an unperturbed conformal field theory* [3]. In particular, at a given cutoff Λ , one induces terms in the Wilsonian action of the form

$$\Delta S_\Lambda = \int d^d x d^d y \gamma_{ab}(x - y; \Lambda) \mathcal{O}_a(x) \mathcal{O}_b(y) \quad (2.4)$$

where \mathcal{O}_a correspond to single-trace operators dual to supergravity fields. The kernel γ is nonlocal over spacetime distances of order Λ^{-1} . In the holographic picture, these operators have a clear interpretation [3]. If one excites the bulk in the “infrared” region $r < r_\Lambda$, these excitations can propagate out to the region $r > r_\Lambda$. The induced term (2.4) acts precisely to describe the transfer of these modes between the IR and UV regions on time scales of order Λ^{-1} . In other words, they take care of the fact that the IR region comprises an open quantum system.

This story was a significant inspiration for the present work where we are trying to characterize the open nature of the effective IR description of interacting theories. However, the relationship between this holographic cutoff and any factorization of the Hilbert space is an open question (see for example [21] for a discussion). In large- N vector models dual to higher-spin theories in anti-de Sitter space, the associated cutoff appears to be a point-splitting cutoff on gauge-invariant bilocal operators [22].

2.3 Hamiltonian renormalization

There is a well-developed literature on Hamiltonian approaches to renormalization; we give a brief summary in order to make clear how our own approach differs.

Wilson’s pioneering work [12, 23], applied first to a model of pion-nucleon scattering and later to the Kondo problem, implemented on a successive diagonalization of degrees of freedom with a hierarchy of energy scales. In these models the degrees of freedom of some quantum field are coupled through a localized defect. At each step one diagonalizes the Hamiltonian of the high energy degrees of freedom coupled

to the defect, and works in the ground states of these degrees of freedom. This diagonalization mixes the (iso)spin states of the defect with excitations of the high-energy modes of the quantum field: at each step, the low-energy spin degree of freedom becomes more delocalized.

Variants for interacting quantum fields (without a defect) can be found in, for example, [13, 24]. In [13] one removes divergences by making a transformation to “band-diagonal” form in which the Hamiltonian has no matrix elements between states with an energy difference larger than some value. In [24] one implements partial diagonalization of the Hamiltonian by removing only the matrix elements between the IR band and the high energy degrees of freedom.

As in Wilson’s work, the goal of the Hamiltonian approach to renormalization is to extract the spectrum. To do so, we reorganize the theory in terms of effective low-energy degrees of freedom where the original low-frequency components of the Hilbert space are appropriately dressed by the high-frequency components so as to partially diagonalize the Hamiltonian between the UV and the IR. If one is studying thermodynamics at low temperatures, or the dynamics of quasiparticles built from the renormalized variables, the low-energy Hilbert space can be treated as a closed quantum system. Similarly, such an approach is also appropriate for S-matrix elements of well-separated particles. In this case, the initial state lies in the low-energy Hilbert space, and the final state will as well. In this way, if one studies the scattering into final states of well-separated particles (or low-energy bound states), one can work entirely within the closed quantum system of the low-energy Hilbert space.

The calculations we describe in this paper, however, assume that measuring devices couple to the bare variables with some finite spatial resolution, and that measurements are made at finite time. In this, low- and high-momentum modes cannot be easily separated, and thus the measurable degrees of freedom form an open quantum system.

2.4 Born-Oppenheimer approximation

Our treatment of long-wavelength modes is closest to the Born-Oppenheimer approximation. In textbook form [25, 26], one separates the quantum mechanical degrees of freedom into “fast variables” Y and “slow variables” x . To implement the approximation, one considers the case that eigenstates of x form a (possibly overcomplete) basis of the Hilbert space described by the slow degrees of freedom, and considers Hamiltonians of the form

$$H = H_x + H_Y(x) \tag{2.5}$$

For example, x could be the positions of heavy nuclei, and Y the positions of electrons moving in the backgrounds of these nuclei.

Consider $H_Y(x)$ for x a c -number. The Hilbert space of the “fast” degrees of freedom can be written in eigenstates $|n, x\rangle$

$$H_Y(x)|n, x\rangle = E_n(x)|n, x\rangle \quad (2.6)$$

Let $\Delta E_x, \Delta E_Y(x)$ be the gap between eigenvalues of $H_x, H_Y(x)$. The simplest version of the Born-Oppenheimer approximation works when $\Delta E_Y \sim E_n(x) - E_m(x) \gg \Delta E_x$, for all x where the wavefunction of the slow degrees of freedom has appreciable support. Let $E_0(x)$ be the instantaneous ground state. One can write the general wavefunction as:

$$|\Psi(t)\rangle = \int dx \psi(x, t) |x\rangle_x |0, x\rangle_Y + \sum_{n>0} \int dx \delta\psi_n(x, t) |x\rangle_x |n, x\rangle_Y. \quad (2.7)$$

To lowest order in the Born-Oppenheimer approximation $\Delta E_x/\Delta E_Y \ll 1$, the leading $n = 0$ term satisfies the time-dependent Schrödinger equation with effective Hamiltonian

$$H_{eff} = H_x + H_{Berry} + E_0(x) \quad (2.8)$$

where H_{Berry} are the additional terms induced by Berry’s phase [26–28].³ Note that in this approximation, the reduced density matrix

$$\rho_{IR} = \text{tr}_Y \int dx dy \psi_0(x) \psi_0^*(y) |x\rangle \langle y| \text{tr}_Y |x, 0\rangle \langle y, 0| \quad (2.9)$$

is a mixed density matrix, so long as $\mathcal{F}(x, y) = \text{tr}_Y |x, 0\rangle \langle y, 0|$ is not factorizable in x and y . Nonetheless, its evolution is unitary, with Hamiltonian H_{eff} , in this approximation. The failure of unitarity – that is, the status of the IR degrees of freedom as an *open* quantum system – will appear at higher orders in the Born-Oppenheimer approximation, for finite-time processes. This has been discussed in the classical limit of the IR degrees of freedom in [30, 31]; corrections to the leading adiabatic limit lead to friction and dissipation.

This framework is essentially what we desire. However, we will be interested in the more general case of systems for which the coupling between IR and UV degrees of freedom cannot be simply expressed in terms of an IR operator which can be diagonalized. An example of this is the Hamiltonian for two coupled spins,

$$H = -\mu_L B S_L^z - \mu_H B S_H^z + \lambda \vec{S}_L \cdot \vec{S}_H \quad (2.10)$$

where $\mu_L \ll \mu_H$, and the total spin $\vec{S}_L^2/\hbar^2 = j_L(j_L + 1)$ is not too large. There is no basis which diagonalizes \vec{S}_L . On the other hand, for $j_L \gg 1$, or for long-wavelength modes in a spin chain, there is a semiclassical limit in which the spin can be treated as a semiclassical variable.

³In the case that there are N near-degenerate eigenstates with energies close to E_0 , ψ_0 is replaced by an N -component wavefunction, with a non-Abelian $U(N)$ Berry’s phase [29].

3. Dynamics of ρ_{IR} in perturbation theory

We return to the framework outlined in the introduction. We will work with a specific set of assumptions:

1. *The interaction λV can be treated perturbatively.* In perturbation theory starting from Fock space there is a tight connection between momentum and energy. We should note that perturbation theory for the density matrix itself can fail at long times, due to secular terms in the perturbative expansion [32].
2. *The eigenvalue spacing for H_{UV} , H_{IR} can be characterized by scales $\Delta E_{UV} \gg \Delta E_{IR}$.* This is a simplification. In general, we expect a local system to have a nested hierarchy of energy levels, corresponding to different momentum modes of the fundamental fields [12].

In addition we will work with two specific initial conditions: first, following much of the literature on open quantum systems, we will consider initial states for which the UV and IR degrees of freedom are not entangled so that $\rho_{IR}(0)$ is a pure state. These have a master equation which is local in time. We will then discuss the case in which the IR and UV are entangled and $\rho_{IR}(0)$ is a mixed state.

3.1 Initial state without UV-IR entanglement

As a first example, we consider $|\Psi(0)\rangle = |\psi_{IR}\rangle|\bar{u}\rangle$, where u labels eigenstates of H_{UV} ; \bar{u} is some particular state, possibly but not necessarily the ground state, while $|\psi_{IR}\rangle$ is taken to be some arbitrary state in \mathcal{H}_{IR} . This initial condition can come from an “interaction quench”, in which λ is suddenly turned on at $t = 0$. The entanglement of spatial regions after a quench has been well-studied, beginning with the pioneering work of [33–35].

In general, when the initial state of a coupled system and environment is factorized between the two, the reduced density matrix of the system satisfies a master equation which is local in time (see [5, 6, 36] for discussion and references):

$$\begin{aligned} i\hbar\partial_t\rho(t) &= [H_{eff}(t), \rho(t)] + i\{A(t), \rho(t)\} + \gamma[\rho(t)] \\ &\equiv [H_{eff}, \rho(t)] + \Gamma[\rho^{(0)}(t)] \end{aligned} \tag{3.1}$$

Γ labels the non-Hamiltonian part of the master equation for ρ , with

$$\begin{aligned} A(t) &= -\frac{1}{2}\sum_k h_{kl}(t)L_l^\dagger(t)L_k(t) \\ \gamma[\rho] &= i\sum_k h_{kl}(t)L_k(t)\rho(t)L_l^\dagger(t) \end{aligned} \tag{3.2}$$

where L_k are some set of operators that can depend on the initial state of the UV degrees of freedom but act on the IR, and h_{kl} is a Hermitian matrix.⁴ This is almost the Kossakowski-Lindblad equation for Markovian dynamics [7, 8]. However, Markovian dynamics requires that the eigenvalues of h be positive; we will find that this condition fails in general.⁵ Thus (3.1) can be non-Markovian even if it looks local in time, because there can be history dependence hidden in the operators A and γ ; this is diagnosed by the positive definiteness of h_{ij} [5, 6, 36].

We will construct (3.1) to second order in perturbation theory, using the fact that the finite-time evolution of ρ has a Kraus representation (see for example [5, 36]):

$$\rho(t) = \sum_{\alpha} K_{\alpha}(t) \rho(0) K_{\alpha}^{\dagger}(t) \quad (3.3)$$

in terms of some operators K_{α} that can be derived from the time evolution.⁶ In our example, the density matrix $\sigma(t) = |\Psi(t)\rangle\langle\Psi(t)|$ in the full Hilbert space $\mathcal{H}_{IR} \times \mathcal{H}_{UV}$ satisfies unitary evolution:

$$\sigma(t) = U(t) \sigma(0) U^{\dagger}(t) , \quad (3.4)$$

where

$$U(t) = e^{-i(H_{IR}+H_{UV})t} T e^{-i\lambda \int_0^t dt' V_I(t')} , \quad (3.5)$$

V_I is the perturbation in the interaction picture, and T is the time-ordering operator. The time evolution of $\rho(t)$ in our case is:

$$\begin{aligned} \rho(t) &= \text{Tr}_{UV} \sigma(t) = \text{Tr}_{UV} U(t, 0) \sigma(0) U^{\dagger}(t, 0) \\ &= \sum_u \langle u | U(t, 0) | \bar{u} \rangle | IR \rangle \langle IR | \langle \bar{u} | U^{\dagger}(t, 0) | u \rangle \\ &= \sum_u \langle u | U(t, 0) | \bar{u} \rangle \rho(0) \langle \bar{u} | U^{\dagger}(t, 0) | u \rangle \end{aligned} \quad (3.6)$$

so that the Kraus operators can be taken to be

$$K_u = \langle u | U(t, 0) | \bar{u} \rangle \quad (3.7)$$

⁴Such state dependence is not usually discussed in Wilsonian renormalization: in particle physics examples, one is usually assuming that the UV theory is in the ground state. More generally, the Born-Oppenheimer discussion above shows that even the effective Hamiltonian (2.8) depends on the state of the UV modes.

⁵In the Kossakowski-Lindblad equation, following from the assumption that the time-evolution of ρ is described by a completely positive dynamical semigroup, h, L are time-independent. However, a more general definition of Markovian includes divisible dynamical maps [37], in which h_k, L_k can be time-dependent, but the eigenvalues of h remain positive.

⁶The Kraus representation guarantees that the map $\rho(0) \rightarrow \rho(t)$ is “completely positive”. This representation is possible when the initial state is disentangled between the IR and the UV. For intermediate times t' , the state will be entangled, and the map from $\rho(t') \rightarrow \rho(t)$ will not be completely positive. This may include t' arbitrarily close to t , as diagnosed by the non-positivity of the eigenvalues of h ; this non-positivity for infinitesimal time evolution means that the evolution will not be Markovian, as entanglement has been generated.

with u indexing UV degrees of freedom.

We can rewrite the master equation (3.1,3.2) in terms of the Kraus operators. Perturbatively,

$$\begin{aligned}\rho(t) &= \rho^{(0)}(t) + \lambda \rho^{(1)}(t) + \lambda^2 \rho^{(2)}(t) + \dots \\ H_{eff} &= H_{IR} + \lambda H_{eff}^{(1)} + \lambda^2 H_{eff}^{(2)} + \dots \\ A &= \lambda A^{(1)} + \lambda^2 A^{(2)} + \dots \\ \gamma &= \lambda \gamma^{(1)} + \lambda^2 \gamma^{(2)} + \dots\end{aligned}\tag{3.8}$$

$\rho(t)$ can be calculated from the Kraus representation by expanding K_u in perturbation. We find to $\mathcal{O}(\lambda^2)$:

$$\begin{aligned}K_{\bar{u}} &= \exp \{-i (H_{eff} + iA) t\} \\ \gamma &= i \partial_t \sum_{u \neq \bar{u}} K_u(t) \rho(0) K_u^\dagger(t)\end{aligned}\tag{3.9}$$

where $K_{\bar{u}}$ is a partial matrix element from transitions between an initial UV state \bar{u} to itself; γ controls transitions out of \bar{u} into other UV states, while A describes the associated loss of unitarity in the subspace $|\bar{u}\rangle \otimes \mathcal{H}_{IR}$.

Since $K_{u \neq \bar{u}}$ is nonvanishing only at $\mathcal{O}(\lambda)$ and higher, $\gamma = \lambda^2 \gamma^{(2)} + \dots$. The form of the time-local master equation (and direct computation) shows that $A = \lambda^2 A^{(2)} + \dots$. Thus, to first order we get simply a correction to the effective Hamiltonian:

$$H^{(1)} = \langle \bar{u} | V | \bar{u} \rangle\tag{3.10}$$

At second order, $A^{(2)}, \gamma^{(2)}$ can be written in the form (3.2). We take k, l to run over $(u \neq \bar{u}, a \in \{1, 2\})$, set $h_{u1, u2} = \frac{1}{\tau^*}$, $h_{u2, u1} = 1/\tau$, and define

$$L_{u,1} = \tau \langle u | V | \bar{u} \rangle\tag{3.11}$$

$$L_{u,2} = \int_0^t dt' \langle u | V_I(t' - t) | \bar{u} \rangle ,\tag{3.12}$$

where τ is an arbitrary complex constant with dimensions of time, chosen so that L_{u1} and L_{u2} have the same dimension, and V_I is the interaction in the interaction picture. The effective Hamiltonian at this order can also be written in terms of $L_{u,i}$:

$$H_{eff}^{(2)} = -\frac{i}{2} \sum_{u \neq \bar{u}} h_{u1, u2} \left(L_{u1}^\dagger L_{u2} - L_{u2}^\dagger L_{u1} \right)\tag{3.13}$$

A natural question is whether the evolution, packaged in this form, is Markovian. In general it is not (nor do we expect it to be). Assume the particularly simple case that V commutes with H_{IR} but not with H_{UV} (when $[V, H_{UV}] = 0$, Γ vanishes). A

short calculation gives

$$\begin{aligned}
& \sum_{u \neq \bar{u}} h_{ui,uj} \left(L_{ui} \rho(t) L_{uj}^\dagger - \frac{1}{2} \left\{ L_{ui}^\dagger L_{uj}, \rho \right\} \right) \\
&= \sum_{u \neq \bar{u}} \frac{2 \sin(E_{u\bar{u}} t)}{E_{u\bar{u}}} \left(\langle u|V|\bar{u} \rangle \rho(t) \langle \bar{u}|V^\dagger|u \rangle - \frac{1}{2} \left\{ \langle \bar{u}|V^\dagger|u \rangle \langle u|V|\bar{u} \rangle, \rho(t) \right\} \right)
\end{aligned} \tag{3.14}$$

where $E_{u\bar{u}} = E_u - E_{\bar{u}}$, and E_u is the energy of $|u\rangle$ with respect to H_{UV} . In general, if $E_{u\bar{u}} \geq \Delta E_{UV}$ for $u \neq \bar{u}$, the sin term will lead to oscillations at the scale ΔE_{UV} . Since the matrix elements in the sum are therefore not positive definite, we know on general grounds [5, 36] that the evolution is not Markovian.

Further insight into when the time evolution of ρ is Markovian comes by rewriting $H^{(2)}$, $A^{(2)}$ and $\gamma^{(2)}$ in terms of correlation functions of the UV parts of V . This representation emerges naturally in the path integral formalism, so we will discuss it further in §4.

3.2 Initial state with UV-IR entanglement

The case in which the initial state is entangled between the UV and IR is harder to characterize. (See sec. 4 of [5] for a preliminary discussion of this case.) Let us consider the specific initial state

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} (|\chi\rangle|u_1\rangle + |\zeta\rangle|u_2\rangle) \tag{3.15}$$

where $|u_{1,2}\rangle$ are eigenstates of H_{UV} with eigenvalues $E_{1,2}$, and $|\chi\rangle, |\zeta\rangle$ are states in \mathcal{H}_{IR} which we will take to be linearly independent. The initial density matrix is

$$\rho(0) = \frac{1}{2} (|\chi\rangle\langle\chi| + |\zeta\rangle\langle\zeta|) \tag{3.16}$$

As we will see, the complication will arise because each term will evolve differently, in a fashion dependent on the UV eigenstates they are coupled to. At zeroth order in λ , the density matrix is simply

$$\begin{aligned}
\rho^{(0)}(t) &= \frac{1}{2} e^{-iH_{IR}t} (|\chi\rangle\langle\chi| + |\zeta\rangle\langle\zeta|) e^{iH_{IR}t} \\
&= \frac{1}{2} (|\chi(t)_I\rangle\langle\chi(t)_I| + |\zeta(t)_I\rangle\langle\zeta(t)_I|)
\end{aligned} \tag{3.17}$$

where $|\psi(t)_I\rangle = e^{-iH_{IR}t}|\psi(0)\rangle$. $\rho^{(0)}(t)$ evolves by Hamiltonian evolution, $i\hbar\partial_t\rho^{(0)}(t) = [H_{IR}, \rho^{(0)}(t)]$.

At first order in λ , a calculation identical to that of the previous section yields:

$$\begin{aligned}
i\hbar\partial_t\rho^{(1)}(t) &= [H_{IR}, \rho^{(1)}(t)] \\
&+ [V_{11}, |\chi(t)_I\rangle\langle\chi(t)_I| + [V_{22}, |\zeta(t)_I\rangle\langle\zeta(t)_I|] \\
&+ [V_{12}e^{-iE_{21}t}, |\zeta(t)_I\rangle\langle\chi(t)_I| + [V_{21}e^{iE_{21}t}, |\chi(t)_I\rangle\langle\zeta(t)_I|]
\end{aligned} \tag{3.18}$$

where $V_{ij} = \langle u_i | V | u_j \rangle$, and $E_{21} = E_2 - E_1 \sim E_{UV}$. There is no obvious sense in which the evolution is Markovian.

3.3 Time averaging

Realistic apparatus have limited accuracy in specifying the time that a given measurement takes place. To find the probability of a given outcome, one should average $P(a, t)$ over a time interval determined by an appropriate window function $f_{\delta t}(\tau, t)$ where t is the peak of the window function and δt is the width. A typical example is a Gaussian

$$f_{g, \delta t}(\tau - t) = \frac{1}{\sqrt{\pi} \delta t} e^{-(\tau - t)^2 / \delta t^2} . \quad (3.19)$$

With this normalization, the sum of (1.5) over all possible orthogonal outcomes is equal to 1. Given a time-dependent function $F(t)$, we denote the time average as:

$$\overline{F(t)} = \int d\tau f_{\delta t}(\tau, t) F(\tau) \quad (3.20)$$

In the case of the Gaussian window function, this expression can be written in Fourier space as:

$$\overline{F(t)} = \int \frac{d\omega}{\sqrt{2\pi}} e^{-\omega^2 \delta t^2 / 4} e^{i\omega t} \tilde{F}(\omega) \quad (3.21)$$

As expected, there is a sharp exponential cutoff for $\omega > \delta t^{-1}$.

We will also be interested in the time average of products of functions. In the case of the Gaussian window function this has a simple closed expression as a sum over products of time-averaged functions:

$$\overline{F(t)G(t)} = \overline{F(t)}\overline{G(t)} + \sum_{n=1}^{\infty} \frac{\delta t^{2n}}{2^n n!} \frac{d^n \overline{F(t)}}{dt^n} \frac{d^n \overline{G(t)}}{dt^n} \quad (3.22)$$

If the time variation of F, G is slow compared to δt , with characteristic frequency Ω , then the average of the product is the product of the averages up to corrections of order $\mathcal{O}((\Omega \delta t)^2)$.

In applying this averaging to (3.1), we will consider $\Delta E_{IR} \ll E_c = \delta t^{-1} \ll \Delta E_{UV}$. Thus we will throw away terms in (3.1) which have net UV time dependence, as these will be exponentially suppressed after time averaging. We will, however, keep terms of order $\mathcal{O}[(\Delta E_{IR}/E_c)^k]$.

3.3.1 Pure initial state

The nontrivial time dependence of the terms in (3.1) arises from $L_{u,2}$ in (3.12). If we study a matrix element of L_2 in the basis $|i\rangle$ of IR eigenstates with energies E_i , we find that

$$\langle i | L_{u,2} | j \rangle = \frac{1 - e^{-i(E_{u\bar{u}} + E_{ij})t}}{i(E_{u\bar{u}} + E_{ij})} \langle i | \langle u | V | \bar{u} \rangle | j \rangle \quad (3.23)$$

where $E_{ij} = E_i - E_j$. The first term will, in general, survive time averaging.

We find that to second order, the time-averaged evolution equation for ρ takes the form:

$$i\partial_t \overline{\rho(t)} = [\overline{H_{eff}}, \overline{\rho(t)}] + i \left\{ \overline{A}, \overline{\rho(t)} \right\} + \overline{\gamma(t)}. \quad (3.24)$$

The time-averaged operators are most easily written in the basis of eigenstates of H_{IR} , and are:

$$\overline{H^{(2)}} = -\frac{1}{2} \sum_{u \neq \bar{u}} \sum_{ij} \left[\frac{\langle u, i | V | \bar{u}, j \rangle}{(\Delta E_u + \Delta E_{ij})} \langle \bar{u} | V | u \rangle |i\rangle \langle j| + \frac{\langle \bar{u}, i | V | u, j \rangle}{(\Delta E_u - \Delta E_{ij})} |i\rangle \langle j| \langle u | V | \bar{u} \rangle \right] \quad (3.25)$$

$$\overline{A^{(2)}} = -\frac{1}{2} \sum_{u \neq \bar{u}} \sum_{ij} \left[\frac{\langle u, i | V | \bar{u}, j \rangle}{(\Delta E_u + \Delta E_{ij})} \langle \bar{u} | V | u \rangle |i\rangle \langle j| - \frac{\langle \bar{u}, i | V | u, j \rangle}{(\Delta E_u - \Delta E_{ij})} |i\rangle \langle j| \langle u | V | \bar{u} \rangle \right] \quad (3.26)$$

$$\overline{\gamma^{(2)}} = \sum_{u \neq \bar{u}} \sum_{ij} \left[\frac{\langle u, i | V | \bar{u}, j \rangle}{(\Delta E_u + \Delta E_{ij})} |i\rangle \langle j| \overline{\rho^{(0)}} \langle \bar{u} | V | u \rangle - \frac{\langle \bar{u}, i | V | u, j \rangle}{(\Delta E_u - \Delta E_{ij})} \langle u | V | \bar{u} \rangle \overline{\rho^{(0)}} |i\rangle \langle j| \right] \quad (3.27)$$

These operators can be written in the form (3.2,3.13). If we choose k, l to run over $(u \neq \bar{u}, a \in \{1, 2\})$, set $h_{u1, u2} = 1/\tau^*$, $h_{u2, u1} = 1/\tau$ and set all other values of h to zero, and define

$$\begin{aligned} \bar{L}_{u,1} &= \tau \langle u | V | \bar{u} \rangle \\ \bar{L}_{u,2} &= -i \sum_{ij} \frac{|i\rangle \langle i| \langle u | V | \bar{u} \rangle |j\rangle \langle j|}{\Delta E_u + \Delta E_{ij}}, \end{aligned} \quad (3.28)$$

it is clear that (3.1,3.2) reduces to (3.24) with the operators defined as in (3.25-3.27). We can write $\bar{L}_{u,2}$ in a more basis-independent form by expanding the denominator in a power series in $(E_{ij}/E_{u\bar{u}})$ and noting that $E_{ij}|i\rangle O_{ij}|j\rangle = [H_{IR}, |i\rangle O_{ij}|j\rangle]$:

$$L_{u,2} = -i \frac{\langle u | V | \bar{u} \rangle}{E_{u\bar{u}}} - i \sum_{k=1}^{\infty} \frac{[\dots [V_u, \overbrace{H_{IR}, \dots, H_{IR}}^{k \text{ times}}]]}{E_{u\bar{u}}^{k+1}} \quad (3.29)$$

where $V_u = \langle u | V | \bar{u} \rangle$ is an operator acting on \mathcal{H}_{IR} . The expansion in IR operators of increasingly high dimension weighted by inverse powers of E_{UV} is what we would expect from a good effective field theory.

The leading-order terms in this expansion are:

$$\begin{aligned}
\overline{H^{(2)}} &= -\sum_{u \neq \bar{u}} \frac{V_u^\dagger V_u}{\Delta E_u} - \frac{1}{2} \sum_{u \neq \bar{u}} \left[\frac{V_u^\dagger [V_u, H_{IR}] - [V_u^\dagger, H_{IR}] V_u}{(E_{u\bar{u}})^2} \right] + \mathcal{O}\left(\frac{1}{E_{u\bar{u}}^3}\right) \\
\overline{A^{(2)}} &= -\frac{1}{2} \sum_{u \neq \bar{u}} \frac{[V_u^\dagger V_u, H_{IR}]}{(\Delta E_u)^2} + \mathcal{O}\left(\frac{1}{E_{u\bar{u}}^3}\right) \\
\overline{\gamma^{(2)}} &= \sum_{u \neq \bar{u}} \left[\frac{[V_u, H_{IR}] \rho^{(0)} V_u^\dagger + V_u \rho^{(0)} [V_u^\dagger, H_{IR}]}{(\Delta E_u)^2} \right] + \mathcal{O}\left(\frac{1}{E_{u\bar{u}}^3}\right)
\end{aligned} \tag{3.30}$$

In other words, to leading order in $1/E_{u\bar{u}}$, the evolution of $\bar{\rho}$ is completely Hamiltonian. This is consistent with our discussion of the Born-Oppenheimer approximation in §2.4, and with the results of [30, 31]. Our results display the kind of decoupling that occurs in Wilsonian renormalization: the effects of transitions to excited states of the UV degrees of freedom are suppressed by powers of $1/E_{UV}$.

One may ask whether the time-averaging we have implemented leads to a Markovian master equation. Once again, there is little indication that this happens in general, beyond the leading order in E_{IR}/E_{UV} for which the evolution is Hamiltonian. If we consider the restricted case $[H_{IR}, V_u] = a_u V_u$, $a_u \in \mathbb{R}$, we find $\overline{A^{(2)}} = \overline{\gamma^{(2)}} = 0$, so that the evolution is not only Markovian but Hamiltonian. Outside of this approximation, \bar{L}_{u2} at order $\mathcal{O}(1/E_{u\bar{u}}^2)$ is not proportional to L_{u1} , so that the negative eigenvalue of $h_{ui,uj}$ will contribute to (3.2). To go further we must examine more specific cases.

3.3.2 Entangled initial state

The time averaging of the first-order evolution equation (3.18) is straightforward – we simply drop the final line, which oscillates rapidly at a time scale of order $1/E_{UV}$. The resulting equation is:

$$i\hbar \partial_t \bar{\rho}(t) = [H_{eff,1}, \frac{1}{2} \overline{|\chi(t)\rangle \langle \chi(t)|}] + [H_{eff,2}, \frac{1}{2} \overline{|\zeta(t)\rangle \langle \zeta(t)|}] \tag{3.31}$$

where

$$H_{eff,i} = H_{IR} + \langle u_i | V | u_i \rangle \tag{3.32}$$

This is not a Hamiltonian evolution.

3.4 Examples

We will work through two simple quantum-mechanical examples, in order to build up our intuition for the different possible dynamics of $\rho(t)$.

3.4.1 Coupled spins

The first example we consider is the case of a set of coupled spins, $\mathcal{H}_{IR} = \mathcal{H}_{j_{IR}}$, $\mathcal{H}_{UV} = \oplus_k \mathcal{H}_{j_{UV,k}}$, where \mathcal{H}_j is the $2j + 1$ -dimensional irreducible representation of $SU(2)$ with spin j . We take the Hamiltonian to be:

$$\begin{aligned} H_{IR} &= -\mu_{IR} B S_{IR}^z \\ H_{UV} &= -\sum_{k=1}^M \mu_{UV,k} B S_{UV,k}^z \\ \lambda V &= \lambda \vec{S}_{IR} \cdot \sum_{k=1}^M \vec{S}_{UV,k} \end{aligned} \quad (3.33)$$

where \vec{S} are the usual spin operators, satisfying $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$ and B is a fixed constant (a magnetic field). We can rewrite the interaction term as

$$\lambda V = \lambda \sum_k S_{IR}^z S_{UV,k}^z + \frac{\lambda}{2} \left(S_{IR}^- \sum_k S_{UV,k}^+ + S_{IR}^+ \sum_k S_{UV,k}^- \right) \quad (3.34)$$

where $S_{\pm} = S_x \pm iS_y$ are the raising and lowering operators in the basis of S_z -eigenstates. We write states in the basis $|j_{IR}, m_{IR}\rangle \prod_k |j_{UV,k}, m_{UV,k}\rangle$ where j is the total angular momentum and m the eigenvalue of S_z . It is straightforward to see that the ground state of $H = H_{IR} + H_{UV} + \lambda V$ is independent of λ to all orders in perturbation theory:

$$|0\rangle = |j_{IR}, m = j_{IR}\rangle \prod_k |j_{UV}, m = j_{UV}\rangle \quad (3.35)$$

Thus, it is natural to consider an initial state of the form

$$|\psi(0)\rangle = C_m (S_{-,IR})^{j_{IR}-m} |0\rangle = |j_{IR}, m\rangle \prod_k |j_{UV}, m = j_{UV}\rangle \quad (3.36)$$

which results from perturbing the ground state by an action of the operator $(S_{IR}^-)^{j-m}$.

The terms in the non-time-averaged equation of motion (3.1) to second order are:

$$\begin{aligned} H_{eff} &= H_{IR} - \lambda \hbar \left(\sum_k g_k j_k \right) S_{IR}^z \\ &\quad - \sum_k \frac{2\lambda^2 \hbar^2 g_k^2 j_k}{(\mu_{UV,k} - \mu_{IR})B} (1 - \cos[(\mu_{UV,k} - \mu_{IR})Bt]) S_{IR}^- S_{IR}^+ \\ A^{(2)} &= - \sum_k \frac{\lambda^2 \hbar^2 g_k^2 j_k}{2(\mu_{UV,k} - \mu_{IR})B} \sin[(\mu_{UV,k} - \mu_{IR})Bt] S_{IR}^- S_{IR}^+ \\ \gamma^{(2)} &= \sum_k \frac{2i\lambda^2 \hbar^2 g_k^2 j_k}{(\mu_{UV,k} - \mu_{IR})B} \sin[(\mu_{UV,k} - \mu_{IR})Bt] S_{IR}^+ \rho^{(0)}(t) S_{IR}^- \end{aligned} \quad (3.37)$$

A, γ can be written in the form (3.2) if the indices are expanded to (ki) where k labels the UV oscillators and $i \in \{1, 2\}$; in this case

$$\begin{aligned}
h_{ki,lj} &= \delta_{kl} \begin{pmatrix} 0 & \frac{1}{\tau^*} \\ \frac{1}{\tau} & 0 \end{pmatrix}_{ij} \\
L_{k1} &= \hbar \sqrt{\frac{j_k}{2}} S_{IR}^+ \\
L_{k2} &= \hbar \sqrt{2j_k} \frac{\exp \left\{ -\frac{i(\mu_{UV,k} - \mu_{IR})Bt}{2} \right\} \sin \left[\frac{(\mu_{UV,k} - \mu_{IR})Bt}{2} \right]}{(\mu_{UV,k} - \mu_{IR})B}
\end{aligned} \tag{3.38}$$

This system is in the class discussed at the end of §3.3.1. The non-Hamiltonian terms $A^{(2)}, \gamma^{(2)}$ are rapidly oscillating, and vanish after time averaging. The effective Hamiltonian for the time-averaged equation at $\mathcal{O}(\lambda^2)$ is:

$$\begin{aligned}
\overline{H}_{eff} &= H_{IR} - \lambda \hbar \left(\sum_k g_k j_k \right) S_{IR}^z - \frac{2\lambda^2 \hbar j}{B} \left[\sum_k \frac{g_k^2}{\mu_k - \mu_{IR}} \right] S_{IR}^- S_{IR}^+ + \mathcal{O}(\lambda^3) \\
&= -\tilde{\mu} B S_{IR}^z + \beta (S_{IR}^z)^2 - E_g
\end{aligned} \tag{3.39}$$

where

$$\begin{aligned}
\tilde{\mu} &= \mu_{IR} - \frac{\hbar \lambda}{B} \sum_k g_k j_k - \frac{\hbar^2 \lambda^2}{B} \sum_k \frac{g_k^2 j_k}{\mu_k - \mu_{IR}} \\
\beta &= \frac{\hbar \lambda^2}{B} \sum_k \frac{g_k^2 j_k}{\mu_k - \mu_{IR}} \\
E_g &= -\frac{\hbar^3 j(j+1)}{B} \sum_k \frac{g_k^2 j_k}{\mu_k - \mu_{IR}}
\end{aligned} \tag{3.40}$$

H_{eff} is related to H_{IR} by renormalization of the magnetic moment, coupling β , and vacuum energy.

For no-factorized initial states of the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\psi_{1,IR}\rangle |j_{UV}, m_1\rangle + |\psi_{2,IR}\rangle |j_{UV}, m_2\rangle)$$

the first-order corrections to the time-averaged evolution equations vanish. We will leave this example for future work.

3.4.2 Linear oscillators

Following the work of [38,39], we consider \mathcal{H}_{IR} the Hilbert space of a simple harmonic oscillator, and \mathcal{H}_{UV} a bath of harmonic oscillators, with the Hamiltonian comprising

a linear coupling between them:

$$\begin{aligned}
H_{IR} &= \frac{P^2}{2M} + \frac{1}{2}M\Omega^2 X^2 \\
H_{UV} &= \sum_k \left[\frac{p_k^2}{2m_k} + \frac{1}{2}m_k\omega_k^2 x_k^2 \right] \\
\lambda V &= \sum_k C_k x_k X
\end{aligned} \tag{3.41}$$

where we take $C_l \sim \mathcal{O}(\lambda)$. Of course, this can be solved exactly by a change of variables. However, in the spirit of this paper we are interested in the dynamics of the “bare” variable X , to which we imagine our measuring devices couple.

In order to match what we expect from a quantum field theory calculation such as that outlined in the introduction, we will take the frequencies $\omega_k \gg \Omega$. Furthermore, we will assume that the UV oscillators are in an eigenstate of H_{UV} (such as the ground state) at leading order in perturbation theory. The resulting system then differs from those studied in [38, 39]. Those works consider the oscillators x_k to be some “environment”, with a spectrum designed phenomenologically to model quantum Brownian motion or dissipation. The environment contains oscillators with arbitrary low frequency, to model dissipation of energy and phase coherence into an environment, over time scales long compared to a given experiment. Furthermore, we are most interested in the UV oscillators initially in their ground state – thus, our treatment is closest to the zero-temperature limit of [38, 39]. In this case, some approximations made in those works fail. Finally, we implement time averaging differently, by directly averaging the density matrix over a coarse-graining kernel. The net result is a qualitatively different master equation for the density matrix.

As stated, we assume that at $t = 0$, the UV oscillators are in an energy eigenstate $|\bar{u}\rangle = \prod_\ell |n_\ell\rangle$. In this case, the first-order shift of the Hamiltonian vanishes, because the expectation value of x_ℓ vanishes in energy eigenstates of the harmonic oscillator. At order λ^2 , we could do the computation starting with (3.28). A more interesting expression arises from computing H, A, γ and extracting a new representation of the form (3.2).

The operators in the second-order time-averaged equation (3.24) are:

$$\begin{aligned}
\overline{H}^{(2)} &= -\frac{1}{2} \sum_\ell \frac{(2n_\ell + 1)C_\ell^2}{m_\ell(\omega_\ell^2 - \Omega^2)} \left[X^2 - \frac{\hbar}{2M\omega_\ell} \right] \\
\overline{A}^{(2)} &= -\frac{1}{2} \sum_\ell \frac{(2n_\ell + 1)C_\ell^2}{2Mm_\ell\omega_\ell(\omega_\ell^2 - \Omega^2)} \{X, P\} \\
\overline{\gamma}^{(2)} &= i \sum_\ell \frac{(2n_\ell + 1)C_\ell^2}{2Mm_\ell\omega_\ell(\omega_\ell^2 - \Omega^2)} (P\overline{\rho}^{(0)}(t)X + X\overline{\rho}^{(0)}(t)P)
\end{aligned} \tag{3.42}$$

At this order, the Hamiltonian is changed by a shift in the oscillator frequency and the ground state energy. We can rewrite A, γ in the form (3.2) if we let i run from

1 to 2 and define:

$$\begin{aligned} h_{12} &= h_{21} = \sum_{\ell} \frac{(2n_{\ell} + 1)C_{\ell}^2}{2m_{\ell}(\omega_{\ell}^2 - \Omega^2)} \\ h_{11} &= h_{22} = 0 \end{aligned} \tag{3.43}$$

$$\begin{aligned} L_1 &= X \\ L_2 &= -i \left(X + i \frac{P}{M\omega_{\ell}} \right) \end{aligned} \tag{3.44}$$

Thus the eigenvalues of the h_{ij} matrix are $\pm h_{12}$. As we explained before, the lack of positive definiteness implies that the evolution of ρ is not Markovian beyond the leading order in Ω/ω_{ℓ} .

Before continuing, it is worth comparing the form of our master equation to that of Caldeira and Leggett [38]. This arises when

- x_{ℓ} describes a continuous spectrum of oscillators, for which $\sum_{\ell} C_{\ell}^2 f(\omega_{\ell}) \rightarrow \int d\omega \rho(\omega) C(\omega)^2 f(\omega)$, and $m_{\ell} = m$, with

$$\rho C^2(\omega) = \frac{2m\eta\omega^2}{\pi} \theta(\Lambda - \omega) \tag{3.45}$$

where Λ is some UV cutoff, and η a phenomenologically determined coefficient.

- Furthermore, the oscillators x_{ℓ} are placed at finite temperature T .

In this case they derive a master equation (Equation (5.12) in [38]) which can be rewritten in the form (3.1,3.2) with the indices $i, j \in \{1, 2\}$ and

$$\begin{aligned} h_{12} &= h_{21} = \frac{\eta kT}{\hbar} \\ L_1 &= X \\ L_2 &= X + \frac{i\hbar}{2MkT} P \end{aligned} \tag{3.46}$$

where η is a function of $C(\omega)$ and the UV cutoff. Note the relative factor of i the coefficient of P in L_2 , as well as the additional term X . In general, the master equation is also not Markovian, unless we were to take the limit $kT \rightarrow \infty$, ηkT finite. (*cf.* [6]).

4. Path integral formalism

The time evolution of the reduced density matrix has a path integral formalism going back to Feynman and Vernon [9,40]. As ever, while the results are equivalent to those we arrived at above, the path integral leads to a particularly nice presentation, and

points to an avenue for a systematic computation of higher-order corrections. Related results appear in the literature (see [6] for a discussion and references), but they are worth recalling and discussing with an eye to the physical systems at hand.

Many readers may be familiar with this formalism in the context of quantum Brownian motion [38] and quantum dissipation [39, 41, 42]; the coupled oscillator model (3.41) is a classic example to which this formalism has been applied. As we discussed in §3.4.2, the “bath” of oscillators x_ℓ in these models contains a continuum of oscillators down to low frequencies, with a spectrum designed so that energy is dissipated into the bath without returning to the observed system over the lifetime of the experiment. Furthermore the bath is typically taken to be at finite temperature. In our discussion, the “bath” consists of degrees of freedom with high frequencies, which are generally in the ground state at leading order in λ .

4.1 Brief review: Density matrix evolution and the influence functional

We wish to compute the density matrix arrived at by beginning with some known state $|0\rangle$ of the full system, evolving the system forward in time, and computing the probability that the final IR state is $|\psi\rangle$. This can be written as:

$$P_\psi = \sum_u \langle u | \langle \psi | U(t, 0) | 0 \rangle \langle 0 | U(t, 0)^\dagger | \psi \rangle | u \rangle \quad (4.1)$$

The amplitude $\langle u | \langle \psi | U(t, 0) | 0 \rangle$ can be expressed as a path integral with the boundary conditions at times 0, t integrated against the wavefunction for $|0\rangle$, $\langle u | \langle \psi |$. Similarly, the amplitude $\langle 0 | U(t, 0)^\dagger | \psi \rangle | u \rangle$ would be represented as the complex conjugate of this path integral: when the system enjoys time reversal invariance, this can be described in terms of paths propagating backwards in time. The result is a path integral over paths moving forward then backwards in time, with the UV degrees of freedom at t set equal and summed over. This is reminiscent of the Schwinger-Keldysh formalism for “in-in” expectation values [43–46]: indeed, if one was to sum the above expression over all final states, one would arrive at the path integral expression for such expectation values.

To make this more explicit, consider a system for which \mathcal{H}_{IR} describes the states of a particle with position X , and \mathcal{H}_{UV} the states of a particle with position x . We consider an action of the form:

$$S[x, X] = S_U[\dot{x}, x] + S_I[\dot{X}, X] + \delta S[x, X] \quad (4.2)$$

where we write the interaction δS in the form

$$\delta S[x, X] = -\lambda \sum_a \int_0^t dt' \mathcal{O}_a^{UV} \Phi_a^{IR} = -\lambda \sum_a \int_0^t dt' \mathcal{O}_a[x] \Phi_a[X] \quad (4.3)$$

We are assuming the interaction term depends on the coordinates only and not on the velocities; thus the correction to the action is minus the correction to the

Hamiltonian. We have factored out a small dimensionless parameter $\lambda \ll 1$ to better organize a perturbative treatment of the system. As in the previous section, we also assume that the initial state of the system can be described by a factorized density matrix

$$\rho(x, X; y, Y; t = 0) = \rho_I(X, Y; 0) \rho_U(x, y; 0) \quad (4.4)$$

The density matrix for the IR degrees of freedom X is:

$$\begin{aligned} \rho(X, Y, t) &= \int dX' dY' J(X, Y, t; X' Y', 0) \rho_I(X', Y', 0), \\ J(X, Y, t; X' Y', 0) &= \int_{Y(0)=Y'; X(0)=X'}^{Y(t)=Y; X(t)=X} \mathcal{D}X \mathcal{D}Y e^{\frac{i}{\hbar} S_I[X] - \frac{i}{\hbar} S_I[Y]} \mathcal{F}[X(t), Y(t)], \end{aligned} \quad (4.5)$$

where ρ_I is the initial density matrix for the IR degrees of freedom, and the *influence functional*

$$\begin{aligned} \mathcal{F}[X(t), Y(t)] &= \int dx' dy' dx \rho_U(x', y', 0) \\ &\times \int_{y(0)=y'; x(0)=x'}^{y(t)=x; x(t)=x} \mathcal{D}x \mathcal{D}y e^{\frac{i}{\hbar} S_U[x] - \frac{i}{\hbar} S_U[y] + \frac{i}{\hbar} \delta S[x, X] - \frac{i}{\hbar} \delta S[y, Y]} \end{aligned} \quad (4.6)$$

contains the dependence on the initial state of the UV degrees of freedom, as well as the interactions between the UV and IR degrees of freedom. We will now pass to computing \mathcal{F} to order $\mathcal{O}(\lambda^2)$.

4.2 Perturbation theory for the influence functional

We expand (4.6) to second order in $\delta S[x, X] - \delta S[y, Y]$ to find

$$\mathcal{F}[X, Y] = \mathcal{F}^{(0)} + \lambda \mathcal{F}^{(1)}[X, Y] + \lambda^2 \mathcal{F}^{(2)}[X, Y] \quad (4.7)$$

using the representation (4.3). Up to first order, we find

$$\begin{aligned} \mathcal{F}^{(0)} &= 1 \\ \mathcal{F}^{(1)} &= \frac{-i}{\hbar} \sum_a \int_0^t dt' \langle \mathcal{O}_a^I(t') \rangle (\Phi_a[X] - \Phi_a[Y]) \end{aligned}$$

where

$$\langle \mathcal{O}_a^I(t') \rangle_0 \equiv \text{Tr}_{UV} (\mathcal{O}_a^I(t') \rho_U(0)) = \text{Tr}_{UV} (\mathcal{O}_a^I(t' - t) \rho_U(t)) \equiv \langle \mathcal{O}_a^I(t' - t) \rangle_t \quad (4.8)$$

The superscript I denotes the interaction picture, and $\rho_U(t)$ evolves as

$$i \partial_t \rho_U(t) = [H_{UV}, \rho_U(t)] \quad (4.9)$$

in this expression.

The second order correction is:

$$\begin{aligned}
\mathcal{F}^{(2)} = & \frac{-\lambda^2}{2\hbar^2} \sum_{a,b} \int_0^t dt' dt'' G_{a,b}^F(t', t'') \Phi_a[X(t')] \Phi_b[X(t'')] \\
& \frac{-\lambda^2}{2\hbar^2} \sum_{a,b} \int_0^t dt' dt'' \tilde{G}_{a,b}^F(t', t'') \Phi_a[Y(t')] \Phi_b[Y(t'')] \\
& \frac{+\lambda^2}{\hbar^2} \sum_{a,b} \int_0^t dt' dt'' G_{a,b}^W(t', t'') \Phi_a[X(t')] \Phi_b[Y(t'')] .
\end{aligned} \tag{4.10}$$

where

$$G_{a,b}^F(t', t'') = \text{Tr}_{UV} T(\mathcal{O}_a^I(t') \mathcal{O}_b^I(t'')) \rho_U(0) , \tag{4.11}$$

$$\tilde{G}_{a,b}^F(t', t'') = \text{Tr}_{UV} \tilde{T}(\mathcal{O}_a^I(t') \mathcal{O}_b^I(t'')) \rho_U(0) , \tag{4.12}$$

$$G_{a,b}^W(t', t'') = \text{Tr}_{UV} \mathcal{O}_a^I(t') \mathcal{O}_b^I(t'') \rho_U(0) , \tag{4.13}$$

and \tilde{T} denotes time anti-ordering. Note that the time-ordered two-point function, the anti-time-ordered two-point function, and the Wightman function appear for essentially the same reason that they do in the Schwinger-Keldysh formalism for in-in expectation values. Finally, note that all of the operators should be understood as being in the interaction picture.

At $\mathcal{O}(\lambda^2)$, we can exponentiate the $\mathcal{O}(\lambda)$ term, to arrive at:

$$\mathcal{F} = e^{\lambda \mathcal{F}^{(1)}} \left(1 + \lambda^2 \tilde{\mathcal{F}}^{(2)} \right) + \mathcal{O}(\lambda^3) \tag{4.14}$$

The effect is to shift G^F , \tilde{G}^F , and G^W to G_c^F , \tilde{G}_c^F , G^W , where

$$G_{c;a,b}^{F,W}(t, t') = G_{a,b}^{F,W}(t, t') - \langle O_a(t) \rangle \langle O_b(t') \rangle \tag{4.15}$$

and $\tilde{G}_c^F = (G_c^F)^*$ as before.

4.3 Relating the influence functional to the master equation

If we insert (4.14) into (4.5), $\mathcal{F}^{(1)}$ can clearly be absorbed into a shift in the action of the form

$$\delta S^{(1)}[X] = - \int_0^t dt'_0 \lambda \langle O_a(t'_0) \rangle \Phi_a[X(t'_0)] \tag{4.16}$$

As a shift in the Hamiltonian, this is identical to the result (3.10) derived via operator methods, if the initial state of the UV degrees of freedom is the pure state $|\bar{u}\rangle$.

Next, by taking the time derivative of (4.5) using (4.14), we can write the master equation to this order in terms of UV Green functions:

$$\begin{aligned}
i\partial_t \rho^{(2)}(t) &= [H_{IR}, \rho^{(2)}] + [H^{(1)}, \rho^{(1)}] \\
&\quad + i\lambda \int_0^t d\tau G_{ab;c}^W(\tau, t) \Phi_b(0) \rho^{(0)}(t) \Phi_a(\tau - t) \\
&\quad + i\lambda \int_0^t d\tau G_{ab;c}^W(t, \tau) \Phi_b(\tau - t) \rho^{(0)}(t) \Phi_a(0) \\
&\quad - i\lambda \int_0^t d\tau G_{ab;c}^W(t, \tau) \Phi_a(0) \Phi_b(\tau - t) \rho^{(0)}(t) \\
&\quad - i\lambda \int_0^t d\tau G_{ab;c}^W(\tau, t) \rho^{(0)}(t) \Phi_a(\tau - t) \Phi_b(0)
\end{aligned} \tag{4.17}$$

This can be written in the form (3.1,3.2) if we write $L_k = L_{ai}$, $i \in \{1, 2\}$, and set

$$\begin{aligned}
h_{ai;bj} &= \lambda^2 \delta_{ab} \begin{pmatrix} 0 & \frac{1}{\tau^*} \\ \frac{1}{\tau} & 0 \end{pmatrix} \\
L_{a1} &= \tau \Phi_a(0) \\
L_{a2} &= \int_0^t d\tau G_{ab;c}^W(t, \tau) \Phi_b(\tau - t)
\end{aligned} \tag{4.18}$$

(where again τ is a complex parameter which factors out of the master equation, but is included so that L_{ai} all have the same dimension).⁷ Finally, the operators in (3.1) then become:

$$\begin{aligned}
H^{(2)} &= i \int_0^t d\tau (G_{ab;c}^W(\tau, t) \Phi_a(\tau - t) \Phi_b(0) - G_{ab;c}^W(t, \tau) \Phi_a(0) \Phi_b(\tau - t)) \\
A^{(2)} &= \int_0^t d\tau (G_{ab;c}^W(\tau, t) \Phi_a(\tau - t) \Phi_b(0) + G_{ab;c}^W(t, \tau) \Phi_a(0) \Phi_b(\tau - t)) \\
\gamma^{(2)} &= i \int_0^t d\tau (G_{ab;c}^W(\tau, t) \Phi_b(0) \rho^{(0)}(t) \Phi_a(\tau - t) + G_{ab;c}^W(t, \tau) \Phi_b(\tau - t) \rho^{(0)}(t) \Phi_a(0))
\end{aligned} \tag{4.19}$$

A straightforward calculation shows that these expressions are equivalent to those in §3.

4.4 Physical interpretation of the master equation

Because we are computing time-dependent inclusive probabilities for measurements of operators supported in the IR, the correct object to coarse-grain is not a transition amplitude but the density matrix for the system. This is the context in which it seems

⁷Note that if we were able to assume that $G_{ab}^W \propto \delta(\tau - t)$, these results would be consistent with Eq. (2.4-2.5) of [18].

correct to integrate out the UV variables. In other words, for the types of questions described in the introduction, the correct analog of the Wilsonian effective action is the influence functional \mathcal{F} .

One technical step we have not achieved is a nice implementation of finite temporal resolution directly in the path integral. Second, the expressions for $H^{(2)}$, $A^{(2)}$, and $\Gamma^{(2)}$ above indicate a necessary condition for local, Markovian evolution, namely that G_{ab}^W (the unordered Wightman function) falls off rapidly for $|\tau - t| \geq \delta t$. In typical quantum systems, this requires (see for example [47, 48]):

- The operators \mathcal{O} should have matrix elements between the initial UV state $|u\rangle$ and a set of UV energy levels that are finely spaced by the inverse of a time scale t_P much larger than the scale of the experiment; at scales of order t_P one expects quasiperiodic behavior characteristic of Poincaré recurrences.
- The matrix elements of \mathcal{O} contributing to the correlation function should have a finite width Γ in energy, leading to exponential falloff at a time scale Γ^{-1} . For finite-temperature correlators, where the Boltzmann factor cuts off large-energy states, this may be of order the inverse temperature.

Local correlators alone are not sufficient to guarantee Markovian behavior. As an example, for a Brownian particle coupled to a spectrum of harmonic oscillators at finite temperature T_B [38], correlators fall off on a time scale of order T_B^{-1} . However, even on time scales long compared to T_B^{-1} , the master equation fails to be Markovian up to a term scaling as γ/T_B (see §3.6.2 of [6]), where γ controls the spectral density of the oscillators and sets the time scale for relaxation of the IR system. For zero-temperature dynamics there is even less reason for Markovian dynamics to emerge.

Since our model two-scale systems do not satisfy the above assumptions, we do not expect Markovian behavior. We reiterate that standard treatments of Wilsonian renormalization are not only Markovian but Hamiltonian, because they work with renormalized variables designed so that the Hamiltonian is factorized between low and high energies. However, we can imagine an experiment in which our apparatus couples to the spatially averaged bare variables. Furthermore, we were inspired by the case of holographic renormalization; if the radial cutoff has anything to do with a split between IR and UV degrees of freedom, it is clear from [3] that the IR degrees of freedom in this case are an open quantum system.

5. UV-IR entanglement

Having computed the density matrix, we can ask how entangled the systems become with time. The most robust quantity to compute is the von Neumann entropy

$$S(t) = -\text{Tr} \rho(t) \ln \rho(t) \quad (5.1)$$

This can be difficult to compute in practice. A simpler set of quantities to calculate are the Rényi entropies for $\rho(t)$:

$$S_n(t) = -\frac{\ln \text{Tr} \rho^n(t)}{n-1}. \quad (5.2)$$

If the resulting expression yields a smooth $n \rightarrow 1$ limit, one may use these to compute the von Neumann entropy.

We must take some care computing S in perturbation theory, due to the logarithm. If the unperturbed density matrix has zero eigenvalues and the perturbation is sufficiently generic, we expect the full density matrix to have eigenvalues that scale as λ^p . Thus, there will be terms that scale as $\lambda^p \ln \lambda$ in the von Neumann entropy, and perturbation theory will break down: this fact was discussed in [2]. While the Rényi entropies for fixed integer $n > 1$ can have good analytic expansions in λ , it is straightforward to see that the $\lambda \rightarrow 0$, $n \rightarrow 1$ limits will not commute. For a simple example, consider the density matrix

$$\rho = \begin{pmatrix} 1 - a\lambda & 0 \\ 0 & a\lambda \end{pmatrix} \quad (5.3)$$

For which

$$\begin{aligned} S_n &= -\frac{1}{n-1} \ln [(1 - a\lambda)^n + (a\lambda)^n] \\ &= \frac{1}{n-1} \ln [(1 - a\lambda)e^{(n-1)\ln(1-a\lambda)} + (a\lambda)e^{(n-1)\ln(a\lambda)}] \end{aligned} \quad (5.4)$$

Note that in the cases we are studying, these entropies capture both the degree to which the initial IR density matrix is in a mixed state, as well as any entanglement that arises from time evolution of the coupled system. Therefore, the most interesting question for us is the evolution of these quantities with time. Focusing on the Rényi entropies with integer n (so that we are sure to work with well-defined quantities), we find:

$$\frac{dS_n(t)}{dt} = \frac{in}{n-1} \frac{\text{Tr}[\rho^{n-1}(t)i\partial_t\rho(t)]}{\text{Tr}\rho^n(t)} \quad (5.5)$$

If we insert (3.1), the contributions from H_{eff} will vanish, due to the cyclicity of the trace, so that:

$$\frac{dS_n(t)}{dt} = \frac{in}{n-1} \frac{\text{Tr}[\rho^{n-1}(t)\Gamma(t)]}{\text{Tr}\rho^n(t)} \quad (5.6)$$

Thus we see that the non-Hamiltonian components of the time-evolution specified by Γ in (3.1) are precisely responsible for producing UV-IR entanglement as time passes.

Let us focus on the particular case that the initial IR state is an energy eigenstate $|i\rangle$, and work to $\mathcal{O}(\lambda^2)$. Since Γ is nonvanishing only at $\mathcal{O}(\lambda^2)$, we can evolve $\rho(t)$

with H_{IR} alone, and it will remain pure. Therefore we can replace $\rho^{n-1} \rightarrow \rho$ for $n > 1$. Then using our known expressions for Γ , we find:

$$\begin{aligned}
\frac{dS_n(t)}{dt} &= \frac{2n}{n-1} \sum_{u \neq \bar{u}, \bar{j} \neq \bar{i}} \frac{\sin \omega_{\bar{u}u, \bar{i}\bar{j}} t}{\omega_{\bar{u}u, \bar{i}\bar{j}}} |\langle \bar{u}, \bar{i} | V | u, \bar{j} \rangle|^2 \\
S(t) &= \int_0^t dt' \frac{dS_n(t')}{dt} \\
&= \frac{2n}{n-1} \sum_{u \neq \bar{u}, \bar{j} \neq \bar{i}} \frac{1 - \cos \omega_{\bar{u}u, \bar{i}\bar{j}} t}{\omega_{\bar{u}u, \bar{i}\bar{j}}^2} |\langle \bar{u}, \bar{i} | V | u, \bar{j} \rangle|^2 \\
\bar{S}(t) &= \frac{2n}{n-1} \sum_{u \neq \bar{u}, \bar{j} \neq \bar{i}} \frac{1}{\omega_{\bar{u}u, \bar{i}\bar{j}}^2} |\langle \bar{u}, \bar{i} | V | u, \bar{j} \rangle|^2
\end{aligned} \tag{5.7}$$

The Rényi entropies thus vary on the time scale of the UV degrees of freedom. Note that $S_n(t) \geq 0$ always: thus the time average is nonvanishing and also time-independent (because the oscillations are at the UV timescale and the IR state is eigenstate of the unperturbed Hamiltonian).

6. Conclusions

We have given a very preliminary description of coarse graining in a set of model systems with a particularly simple structure of energy levels, a simple band structure determined by two scales $\Delta E_{UV}, \Delta E_{IR}$. A local quantum field theory, however, will have a nested hierarchy of energy levels, determined by a hierarchy of scales $\Delta E_1 \ll \Delta E_2 \ll \Delta E_3 \cdots \ll \Delta E_N$. In perturbation theory, one would first trace out the bare degrees of freedom characterized by energy scale ΔE_N , and so on down. We expect that the terms H, A, γ in the master equation for ρ will run smoothly under this successive integrating out, just as they do in prior treatments of Hamiltonian renormalization. More generally, it will be important to explicitly compute the master equation for various quantum field theories.

In using the phrase “coarse-graining” in our perturbative treatment, we have been implying that we could assign an energy scale to the degrees of freedom we were tracing out, corresponding to a short distance scale. This makes sense in weakly-coupled quantum field theories, in which the energy and momenta of single quanta are tied together and there is some meaning to these single quanta. In strongly-coupled systems, especially those without quasiparticle excitations, this relation breaks down. It would be interesting to study our coarse-graining in such examples, either analytically or numerically.

6.1 Additional questions

Cosmological perturbations. Primordial non-gaussianities in CMB fluctuations and large-scale structure measure correlations between quantum fluctuations at different

scales, induced by interactions in the inflaton sector. We expect the initial state to have a degree of quantum entanglement between scales, following the discussions in this paper, which should help seed the classical correlations one actually observes.

Some discussion of entanglement between scales during inflation appears in [18]. In this work, the entanglement between short- and long-wavelength modes is used to justify a Lindblad equation describing Markovian evolution for the long-wavelength modes, based on an argument that the Hubble scale sets a natural time scale for the decay of correlations of short-wavelength modes. It would be interesting to perform a more quantitative, first-principles analysis of entanglement between scales in some specific model, following the discussion here. For example, as we have noted, even when correlation functions are essentially local in time, the dynamics of long-wavelength modes can still fail to be Markovian.

Holographic renormalization. In our setup, the evolution equation for ρ is local on scales larger than ΔE_{UV} . This is in accord with the discussion of holographic gauge theories in [3, 14, 15, 22], in which the Wilsonian effective action of a strongly-coupled field theory was nonlocal on the scale of the cutoff, reflecting the propagation of excitations into and back out of the UV region. However, string theory suggests that there are other nongravitational theories in which the time scale over which excitations are supported in the UV becomes arbitrarily large. One example is little string theory – in the holographic dual, massless excitations propagating into the UV region take an infinite time to reach the “boundary”. This is tied to the exponential (Hagedorn) growth of states at high energies in this theory. It would be interesting to explore the dynamics of ρ_{IR} in this setting.

More generally we would like a more precise understanding of the relationship between the framework in this paper and that of Wilsonian renormalization in holographic gauge theories, in which one “integrates out” a section of the geometry [3, 14, 15]. For example, this could cast an interesting light on black hole entropy. There is evidence that bulk quantum corrections to the entanglement entropy of quantum fields between the interior and exterior of a “stretched horizon” outside the black hole are mapped to the Wald entropy of the black hole, using the renormalized gravitational action (see [49, 50] and references therein); and there are conjectures that the full Bekenstein-Hawking/Wald entropy of the black hole can be considered as an entanglement entropy (see for example [51]).

In holographic theories, black holes are dual to high-energy states with thermal behavior. In closed quantum systems, the Eigenstate Thermalization Hypothesis [52, 53] states that a class of quantum operators will have expectation values and correlation functions which appear to be thermal. In many examples, these are local operators supported in a spatial subregion of the system, and the excited quantum state is strongly entangled between the subregion and its complement so that the reduced density matrix looks approximately thermal (see [54] for a recent discussion,

and further references.) Of course, this is not the only way to decompose the Hilbert space such that the state is entangled between the components. The stretched horizon appears at some radius in the AdS-black hole geometry, whose value should be dual to some scale in the field theory dynamics. Studying entanglement between degrees of freedom at different scales could shed light on this system.⁸

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⁸On a related note, Ref. [49] studies the progressive contribution of longer and longer wavelengths of *bulk* fields to the black hole entanglement entropy.

References

- [1] R. Thomale, D. Arovas, and B. A. Bernevig, “Nonlocal order in gapless systems: entanglement spectrum in spin chains,” *Physical review letters* **105** (2010) no. 11, 116805.
- [2] V. Balasubramanian, M. B. McDermott, and M. Van Raamsdonk, “Momentum-space entanglement and renormalization in quantum field theory,” *Phys.Rev.* **D86** (2012) 045014, [arXiv:1108.3568 \[hep-th\]](#).
- [3] V. Balasubramanian, M. Guica, and A. Lawrence, “Holographic Interpretations of the Renormalization Group,” *JHEP* **1301** (2013) 115, [arXiv:1211.1729 \[hep-th\]](#).
- [4] H.-P. Breuer, “Foundations and measures of quantum non-Markovianity,” *Journal of Physics B Atomic Molecular Physics* **45** (Aug., 2012) 154001, [arXiv:1206.5346 \[quant-ph\]](#).
- [5] Á. Rivas, S. F. Huelga, and M. B. Plenio, “Quantum non-Markovianity: characterization, quantification and detection,” *Reports on Progress in Physics* **77** (Sept., 2014) 094001, [arXiv:1405.0303 \[quant-ph\]](#).
- [6] H. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*. OUP Oxford, 2007.
- [7] A. Kossakowski, “On quantum statistical mechanics of non-hamiltonian systems,” *Reports on Mathematical Physics* **3** (1972) no. 4, 247 – 274.
- [8] G. Lindblad, “On the Generators of Quantum Dynamical Semigroups,” *Commun.Math.Phys.* **48** (1976) 119.
- [9] R. Feynman and J. Vernon, F.L., “The Theory of a general quantum system interacting with a linear dissipative system,” *Annals Phys.* **24** (1963) 118–173.
- [10] K. Wilson and J. B. Kogut, “The Renormalization group and the epsilon expansion,” *Phys.Rept.* **12** (1974) 75–200.
- [11] K. Wilson, “The renormalization group and critical phenomena,” *Rev.Mod.Phys.* **55** (1983) 583–600.
- [12] K. G. Wilson, “The Renormalization Group: Critical Phenomena and the Kondo Problem,” *Rev.Mod.Phys.* **47** (1975) 773.
- [13] S. D. Glazek and K. G. Wilson, “Renormalization of Hamiltonians,” *Phys.Rev.* **D48** (1993) 5863–5872.
- [14] I. Heemskerk and J. Polchinski, “Holographic and Wilsonian Renormalization Groups,” *JHEP* **1106** (2011) 031, [arXiv:1010.1264 \[hep-th\]](#).

- [15] T. Faulkner, H. Liu, and M. Rangamani, “Integrating out geometry: Holographic Wilsonian RG and the membrane paradigm,” *JHEP* **1108** (2011) 051, [arXiv:1010.4036 \[hep-th\]](#).
- [16] D. A. Meyer, “Scale decoherence in inhomogeneous potentials,” [arXiv:quant-ph/9805039 \[quant-ph\]](#).
- [17] S. V. Akkelin and Yu. M. Sinyukov, “Entanglement of scales as a possible mechanism for decoherence and thermalization in relativistic heavy ion collisions,” *Phys. Rev.* **C89** (2014) no.~3, 034910, [arXiv:1309.4388 \[nucl-th\]](#).
- [18] C. Burgess, R. Holman, G. Tasinato, and M. Williams, “EFT Beyond the Horizon: Stochastic Inflation and How Primordial Quantum Fluctuations Go Classical,” [arXiv:1408.5002 \[hep-th\]](#).
- [19] D. Neill, “The Edge of Jets and Subleading Non-Global Logs,” [arXiv:1508.07568 \[hep-ph\]](#).
- [20] M. E. Peskin and D. V. Schroeder, “An Introduction to quantum field theory,”.
- [21] V. Balasubramanian, B. D. Chowdhury, B. Czech, J. de Boer, and M. P. Heller, “A hole-ographic spacetime,” *Phys.Rev.* **D89** (2014) 086004, [arXiv:1310.4204 \[hep-th\]](#).
- [22] E. Mintun and J. Polchinski, “Higher Spin Holography, RG, and the Light Cone,” [arXiv:1411.3151 \[hep-th\]](#).
- [23] K. G. Wilson, “Model Hamiltonians for Local Quantum Field Theory,” *Phys.Rev.* **140** (1965) B445–B457.
- [24] G. Alexanian and E. Moreno, “On the renormalization of Hamiltonians,” *Phys.Lett.* **B450** (1999) 149–157, [arXiv:hep-th/9811158 \[hep-th\]](#).
- [25] A. Messiah, *Quantum Mechanics*. Dover books on physics. Dover Publications, 1999.
- [26] J. Moody, A. D. Shapere, and F. Wilczek, “Adiabatic Effective Lagrangians,” in *Geometric Phases in Physics*, A. D. Shapere and F. Wilczek, eds. World Scientific, 1989.
- [27] M. V. Berry, “Quantal phase factors accompanying adiabatic changes,” *Proc.Roy.Soc.Lond.* **A392** (1984) 45–57.
- [28] B. Simon, “Holonomy, the quantum adiabatic theorem, and Berry’s phase,” *Phys.Rev.Lett.* **51** (1983) 2167–2170.
- [29] F. Wilczek and A. Zee, “Appearance of Gauge Structure in Simple Dynamical Systems,” *Phys.Rev.Lett.* **52** (1984) 2111–2114.

- [30] L. D'Alessio and A. Polkovnikov, "Emergent Newtonian dynamics and the geometric origin of mass," *Annals of Physics* **345** (June, 2014) 141–165, [arXiv:1309.6354](#) [`cond-mat.stat-mech`].
- [31] L. D'Alessio, Y. Kafri, and A. Polkovnikov, "Negative mass corrections in a dissipative stochastic environment," *ArXiv e-prints* (May, 2014) , [arXiv:1405.2077](#) [`cond-mat.stat-mech`].
- [32] C. H. Fleming and N. I. Cummings, "Accuracy of perturbative master equations," *Physical Review E* **83** (2011) no. 3, 031117.
- [33] P. Calabrese and J. L. Cardy, "Evolution of entanglement entropy in one-dimensional systems," *J.Stat.Mech.* **0504** (2005) P04010, [arXiv:cond-mat/0503393](#) [`cond-mat`].
- [34] P. Calabrese and J. L. Cardy, "Time-dependence of correlation functions following a quantum quench," *Phys.Rev.Lett.* **96** (2006) 136801, [arXiv:cond-mat/0601225](#) [`cond-mat`].
- [35] P. Calabrese and J. Cardy, "Quantum Quenches in Extended Systems," *J.Stat.Mech.* **0706** (2007) P06008, [arXiv:0704.1880](#) [`cond-mat.stat-mech`].
- [36] H.-P. Breuer, "Foundations and measures of quantum non-Markovianity," *Journal of Physics B: Atomic, Molecular and Optical Physics* **45** (2012) no. 15, 154001.
- [37] E.-M. Laine, J. Piilo, and H.-P. Breuer, "Measure for the non-Markovianity of quantum processes," *Phys. Rev. A* **81** (June, 2010) 062115, [arXiv:1002.2583](#) [`quant-ph`].
- [38] A. Caldeira and A. Leggett, "Path integral approach to quantum Brownian motion," *Physica* **121A** (1983) 587–616.
- [39] A. Leggett, S. Chakravarty, A. Dorsey, M. P. Fisher, A. Garg, *et al.*, "Dynamics of the dissipative two-state system," *Rev.Mod.Phys.* **59** (1987) 1–85.
- [40] R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals: Emended Edition*. Dover Publications, Incorporated, 2012.
- [41] A. Caldeira and A. Leggett, "Influence of dissipation on quantum tunneling in macroscopic systems," *Phys.Rev.Lett.* **46** (1981) 211.
- [42] A. Caldeira and A. Leggett, "Quantum tunneling in a dissipative system," *Annals Phys.* **149** (1983) 374–456.
- [43] J. S. Schwinger, "Brownian motion of a quantum oscillator," *J.Math.Phys.* **2** (1961) 407–432.
- [44] P. M. Bakshi and K. T. Mahanthappa, "Expectation value formalism in quantum field theory. 1.," *J.Math.Phys.* **4** (1963) 1–11.

- [45] P. M. Bakshi and K. T. Mahanthappa, “Expectation value formalism in quantum field theory. 2.,” *J.Math.Phys.* **4** (1963) 12–16.
- [46] L. Keldysh, “Diagram technique for nonequilibrium processes,” *Zh.Eksp.Teor.Fiz.* **47** (1964) 1515–1527.
- [47] J. Barbon and E. Rabinovici, “Very long time scales and black hole thermal equilibrium,” *JHEP* **0311** (2003) 047, [arXiv:hep-th/0308063](#) [[hep-th](#)].
- [48] L. Dyson, M. Kleban, and L. Susskind, “Disturbing implications of a cosmological constant,” *JHEP* **0210** (2002) 011, [arXiv:hep-th/0208013](#) [[hep-th](#)].
- [49] T. Jacobson and A. Satz, “Black hole entanglement entropy and the renormalization group,” *Phys. Rev.* **D87** (2013) no.~8, 084047, [arXiv:1212.6824](#).
- [50] J. H. Cooperman and M. A. Luty, “Renormalization of Entanglement Entropy and the Gravitational Effective Action,” [arXiv:1302.1878](#) [[hep-th](#)].
- [51] E. Bianchi and R. C. Myers, “On the Architecture of Spacetime Geometry,” *Class.Quant.Grav.* **31** (2014) no.~21, 214002, [arXiv:1212.5183](#) [[hep-th](#)].
- [52] J. M. Deutsch, “Quantum statistical mechanics in a closed system,” *Phys. Rev. A* **43** (Feb, 1991) 2046–2049.
- [53] M. Srednicki, “Chaos and quantum thermalization,” *Phys. Rev. E* **50** (Aug, 1994) 888–901.
- [54] R. Nandkishore and D. A. Huse, “Many body localization and thermalization in quantum statistical mechanics,” [arXiv:1404.0686](#) [[cond-mat.stat-mech](#)].